WHY STUDENTS FAIL IN MATHEMATICS.

By HELEN A. MERRILL.

This subject was suggested some weeks ago, at a meeting of the Council of this Association, as one in which both school and college teachers are likely to be interested. There is a certain advantage in directing the attention occasionally to facts with which all are perfectly familiar, particularly if those same facts involve problems which are never completely solved, but which must be faced anew with each new set of students that enters our classrooms. This is my reason for undertaking today to open a discussion of a subject with which all of us may well be equally familiar. I am further encouraged by a remark made last fall by Professor (now Major) Julian Coolidge, who told an audience that he was addressing on a rather elementary subject, that while he had often been bored by too difficult mathematical discussions, he had never yet been bored by too easy ones, a sentiment in which every hearer probably concurred.

This subject enjoys the marked advantage of needing no definitions of terms, no proof of the claim which it implies. We all know what failures in mathematics, technically so called, mean, and that students make these failures.

Statistics as to the number of failures in mathematics and other subjects have been published in many school and college reports, in educational journals, etc. Into this side of the question I do not propose to enter, though it is one that interests me greatly, and I have for many years collected averages and percentages for

private use only. I assume without attempt at proof that the average of failures is rather high, probably higher than in any other subject, and suggest possible reasons, which may well be added to by others, with the hope that remedies also may be suggested. For there is little reason for scrutinizing the dark side of any subject, unless some means be thereby found of brightening that side, not by whitewashing it, but by removing the dust and stains of neglect and ignorance.

I propose to speak under four general heads of the main reasons for failures in the study of mathematics, asking successively how much of the failure may be due to the subject itself, to those who teach it, to parents and friends of students, and finally to the students themselves.

That mathematics is a difficult subject no one can deny. It demands, among other things, accuracy of thought and statement, definite mental concepts, connected thinking, a fair memory, quickness to recognize relations between forms and numbers, power of generalization, a willingness to work hard. The abstractness of much of the work renders it difficult to most minds, and obnoxious to some. The ideal elements that are the object of study seem remote and unreal. This troubles those who like at each step to feel the solid ground under their feet, while those who revel in flights of fancy are continually being pulled to earth by the insistent *why* and the demand for obedience to the most important laws of logic. There is a new language to be learned, new symbols to be manipulated, new elements to be dealt with, in some cases a queer upsetting and rearranging of one's mental processes. The ability to reason about certain subjects seems, oddly enough, not always to ensure that ability in mathematics, the thought processes are different, highly specialized, and subject to rigid rules, the topics of thought seem remote from everyday thinking and experience, impersonal and detached, and at certain ages students demand intense human interests, the object of the work as a whole is not readily seen. We sometimes hear the ability to put 2 and 2 together referred to as a rather rare gift, and the thought processes required to draw even a very simple conclusion from given algebraic or geometric premises is more complicated than we sometimes think. In a book by Benchara Branford entitled

" A Study of Mathematical Education," there is an interesting
and instructive account of the author's methods of work in geom-
etry with quite young children. One intelligent child could
readily find by superposition that the triangle A was exactly
equal to the triangle B and to the triangle C, but did not suspect
that this discovery involved any relation between B and C. In
fact the teacher was unable to make this relation clear by any
method but the superposition of B and C.

The very definitions of mathematics that have been suggested
often emphasize the difficulty of approach for the average youth-
ful mind. The abstract quality of the subjects studied is evi-
dent in all, while in some of them is stated or implied that this
science involves the drawing of necessary conclusions. But
these conclusions are in many instances *necessary* only to the
mind that is sufficiently trained in drawing them to feel their
necessity. To the bewildered student, groping painfully from
point to point, the triumphant *therefore* does not always mean
that the goal is necessarily attained; in fact, we may at times
strongly suspect that a *because* or an *and* would be accepted with
equal resignation and seem quite as plausible.

These are some of the many characteristics that make mathe-
matics hard. Of course it is hardly necessary to say that many
of these very characteristics make it attractive and fascinating
to some types of mind. The fact that it is abstract and imper-
sonal, that it requires effort and compels concentration has made
it a welcome refuge to minds distracted by the perplexities of
everyday life. Probably every one of us knows the joy of
losing ourselves and our cares completely in a problem that
taxes our minds. How shall we learn to appreciate the con-
fusion of minds lost in a sea of vague, elusive ideas, never quite
grasped, finding in even the simplest piece of work pitfalls
whose existence we never imagined, reading into what seem to
us plain and obvious statements most extraordinary meanings,
the evolving of which would seem to be a tax to any brain! We
must all have been astounded at the ability of the dull or aver-
age student to misinterpret the simplest statements, an ability
that makes it necessary to look at every question on an exam-
ination paper from all possible, or even impossible, angles.

But these considerations are leading us away from mathe-

matics proper to those who are teaching it, and our second topic
is already before us.

How far can the blame for failures in mathematics be laid
upon the teachers? None of us will claim that teachers are
faultless, nor are we ready to admit meekly that all the fault
lies here.

The elementary mathematics have been taught for so long
that the method, as well as their content, has been quite fully
standardized, and while no satisfactory measurements can be
made, it is probable that mathematics is taught in schools and
colleges at least as well as any other subject. The last issue of
THE MATHEMATICS TEACHER calls attention to what it terms " a
deplorable condition in five states," the statistics showing that
one out of every five high school teachers has had nothing above
normal school training, while some have not even high school
training. New England figures of course show a very different
state of affairs, but many students in New England come from
other states and thus their problem becomes our problem. How-
ever little poor teaching of mathematics there may be, there is
certainly more than there should be. Even when high school
teachers are listed as college graduates, it by no means follows
that they have had more than one year of college mathematics.
Whatever efforts mathematics dpartments may make, refusing to
recommend unfit candidates, and trying to keep students from
entering fields for which they have little preparation, teachers'
agencies and even school superintendents often seem possessed
of the idea that a B.A. implies the ability to teach any high school
subject, and the program of a teacher of Latin or English is
completed by adding a mathematics class, generally in beginning
algebra, for there seems to be a rather widely prevalent notion
that anyone can teach that. It is claimed that this work cannot
be avoided, it is a choice between teaching this class and declin-
ing the position, and urgent advice to refuse to teach any sub-
ject in which one is not at least three years ahead of one's class
is apt to be regarded by principals and teachers alike as imprac-
tical.

I do not mean to claim that all such teaching is a failure, but
I do claim, and that emphatically, that this unhappy fact ac-
counts in part for the attitude of some students toward mathe-

matics. I have been told by some few students that never until they entered college had they been taught by any one who showed any enthusiasm for the subject, and the fact that mathematics has any qualities capable of inspiring enthusiasm was to them a quite new idea. Such students have been cheated of one of their rights—the proper approach to a great subject. To the teaching of mathematics quite as much as to any subject is that homely old saying applicable, that there must be more than a quart of molasses in a jug if you are to get a quart out. Or, to use a different figure, the right guide for young travellers through a wide-stretching land is surely not the one who knows just one path leading only a few miles into the country. Even if he knows that short stretch of the way rather well, how is he to point out to his often slow and reluctant followers the interesting side paths, and occasional far-reaching glimpses.

A great amount of mathematical knowledge does not of course ensure first class teaching, many other elements enter in, but, other things being equal, all the chances of success are on the side of the one who can look at each day's work from many different points of view, who can estimate the value of each part of the subject, not according to its bearing upon the coming examination, but according to its value for mental development and its later usefulness.

When the student's preparation for college is plainly at fault, the lack is apt to be in one of a few directions. Probably there has been insufficient drill on the fundamentals, due either to a failure to appreciate their importance or to lack of time. The school year has grown shorter, and many more subjects are clamoring for the student's attention. Small wonder that teachers everywhere are conscious of the necessity of cutting down to the smallest possible limit the amount of time to be assigned to each topic. It takes a far-sighted and experienced teacher to choose wisely between the topics that may safely be given less consideration, and those that must be drilled into the students' minds until they become second nature, whatever the cost in time may be. Sometimes too elementary a text has been used, and the mental muscles of the student have not been developed. But with the right kind of teacher the text-book is relatively unimportant, and the most up-to-date, all-inclusive,

pedagogically perfect text cannot be guaranteed as fool-proof.

Sometimes the numbers in the class have been so large that the student has not received the individual attention needed. This lack displays itself chiefly in untidy or badly arranged work, which as a rule indicates the need of mental discipline.

Sometimes the student has suffered from being personally conducted over every difficulty, with every possible thorn and stone removed from the path, and assistance rendered at every turn. Such a student is likely to be handicapped in later work, it is hard to throw away the crutches upon which one has been encouraged to lean. One hears with dismay the demand for private tutoring when difficulties appear, and wonders whether the schools are fostering such aids as much as seems to be indicated. When a good part of the fun of mathematics lies in doing one's own thinking, it is unfortunate that any student should fail to find, or at least to be encouraged to find, this pleasure in independent work.

It is quite possible for prospective teachers to spend so long a time in study that they get out of touch with young students The difficulties which they encounter are far removed from our experience, the subjects look so absurdly easy. We are led again to the question which teachers can hardly put to themselves too often:—How shall we learn to see clearly our pupils' difficulties? How can we put ourselves in their place and so learn to forestall their blunderings, and make the subject—not necessarily easy, but clear?

· Any one who is teaching a child to read would do well at times, when progress seems to be slow, to try reading aloud from a book held upside down. The letters and words will probably have a far more familiar look to him than the ordinary page will have to the child. Some such experience is often illuminating; in fact, one of the chief safeguards against impatience or discouragement or lack of sympathy with our students is to give ourselves a chance occasionally to show how stupid we can be. We need to be working on something new and hard all the while that we are engaged in teaching, and, unless our minds are quite exceptional, we shall be taught thereby how easy it is to jump at wrong conclusions, to miss the point of the argument, to fail to see something which we finally discover to be surprisingly obvious.

There is nothing easier than to give instructions, all-inclusive and infallible, for success in teaching. You have probably read these two rules, guaranteed to make any one who follows them successful:

Rule 1.—Know everything there is to be known about the subject.

Rule 2.—Arouse in the mind of each student an eager desire and determination to know all there is to be known about the subject.

Like all really good nonsense, this has a large modicum of common sense in it. Our failures as teachers can probably be classed more or less roughly under two heads—failure to know enough about the subject, its history, its connection with other subjects, its interesting applications, the relative importance of its various subdivisions and the best methods of presenting them; and failure to give ourselves wholeheartedly to our teaching, to enjoy it so thoroughly, to appreciate it so keenly that our students cannot fail to be influenced by our enthusiasm.

No student ought to complete a course in mathematics without the feeling that there must be something in it, without catching a glimpse, however fleeting, of its possibilities, without at least a few moments of pleasure in achievement and insight.

The third reason for failure may be disposed of more briefly. How much have one's family or friends or the outside non-mathematical public to do with this matter? It is often a wonder that the modern boy and girl find time to do any studying. Many parents resent any attempt to have work brought home. Music lessons, dancing lessons, etc., take much time, and young folks are allowed an extraordinary amount of social diversion, with late hours and excitement telling sadly on the next day's work. This however must affect one study about as much as another, and while it doubtless helps to account for some failures, it has no exclusive relation to mathematics. I wish to speak of a certain psychological influence which has a marked effect upon our mathematical students especially.

It is clearly impossible to arrange a scale of hardness in studies such as is used in mineralogical tests. But if the formation of such a scale were attempted, mathematics would probably head most of the lists. Once label a subject *Very hard,* and let that

label be flaunted before the young pupils' sight, and they are handicapped from the start. They magnify every difficulty, are discouraged too easily, accept failures as all but inevitable. This disadvantage works in many ways. Children are pitied for having to work hard examples, they are made to tremble at the very thought of algebra and geometry. If they express any pleasure in the subject they are called grinds, or sharks, or are told "Just wait till you get to radicals." Students who have just finished a course in algebra and geometry delight in terrifying those in the class below them, exaggerating its difficulties, discouraging them from reasonable efforts to succeed by instilling a belief in the futility of such attempts, magnifying the slaughter wrought by examinations, or perhaps declaring that the only way they themselves got through was by committing all the proofs to memory, a tale which can rarely be true, but which is often swallowed with avidity. If it were possible to eliminate from the young minds, that cling so tenaciously to some forms of tradition, this conventional view of mathematics, I believe that we should find pleasure in learning and in teaching mathematics wonderfully increased, and failures in the subject correspondingly diminished. Is there any way in which we can achieve this? It is worth much thought and effort.

The general attitude toward mathematics for girls is peculiarly hampering to them. I often marvel, not that some fail, but that so large a proportion succeed, when fathers and mothers say to me: "It seems such a pity for girls to have to waste their time on mathematics. It is always hard for them, and why should they be obliged to study it?" And then they often add "It isn't as if my daughter would ever have to earn her own living," their theory evidently bring that the only possible reason why a girl should study mathematics is in order to teach it to others, who will have no mortal use for it except to teach it to other poor victims, etc.

A not at all bloodthirsty school principal once remarked: "I could have an excellent school if I could kill off all the parents." Many of us may have had a similar feeling. But it is not the parents only. Why should we blame them, when high educational officials go about the country proclaiming: "Why teach girls algebra?", "Why compel students to take any mathematics beyond arithmetic?"

Here again the fault is in part in the teaching. Have we allowed the work to become too formal, too theoretical, too removed from everyday use? If we could in some way help the students, their parents, the general public, to see what mathematics really is like, and what it can do, would not some of this outcry cease?

Some efforts to bring mathematics up to date are frankly amusing. Whether the substitution of Zeppelins and airships for the old-time fox and greyhound or couriers makes the subject seem more vital, some who are here may be able to say. Letting the pipes which in Hero's days emptied into a cistern now convey gasoline into an automobile tank may make them easier to reckon with. Perhaps the change of name helps the pain.

But if we can in some way help all who have an interest in our students to see that mathematics is of service to them, not alone in terms of dollars and cents, though we readily grant that type of usefulness, but as an aid in gaining powers and qualities that all wish our young folks to acquire, this cause of failure in mathematics will be removed.

And finally, what of the students themselves? How far are they to be blamed for lack of success?

Individual failures are usually to be ascribed to one or more of three lacks—lack of ability, lack of training, lack of effort. The first may be due to the grandparents, the second may be the fault of teachers or parents, the third in the ultimate analysis must come home to the student, and it is the one that counts for the most. A dull or slow student may succeed through hard work, poor preparation may be offset by hard work, but persistent unwillingness to work cannot long be counterbalanced by natural ability or thorough preparation.

Are there any boys and girls who really cannot do mathematics? I presume that there are. In a western high school, which has a fine reputation for its mathematical work, I visited a class composed of boys and girls who, after many attempts on the part of their teachers, and possibly on their own part, seemed quite unable to do the ordinary work. So an attempt was being made to give them very simple and very practical work in arithmetic, for which they might find ready use. It had, however,

proved nearly if not quite as hard for them to get on in this class. It was a small class, but the amount of stupidity registered in their faces would have been a generous allowance for a very large class. They had the look of being quite unequal to any intellectual effort, but they were strong, husky looking boys and girls, and should have been put soon to some form of manual labor, which might perhaps develop their minds along with their bodies.

But boys and girls of mediocre ability can surely do the amount of mathematics ordinarily required of them in school if they have pluck and determination, while it seems likely that students who have sufficient intelligence to warrant their being in college, who are what is known as " college material," can do as much mathematics as is ever required in those colleges which still hold to the prescribed course. Failure in these elementary courses is rarely due to actual inability to do the work. The difficulty is that into the bright lexicon of youth has crept a word that was once supposed not to be found there.

A colleague in our Latin department said to me lately: " I can tell you in one word why students fail in mathematics and in Latin prose composition—*obstinacy*." I repeated this later to a group of students, rather expecting indignant denial, but to my surprise they agreed heartily.

Anticipation of difficulty and the psychological influence already referred to have made them accept failure without a struggle, and, having thus resigned themselves, it is most difficult to arouse them to genuine and determined effort. Their attitude is well expressed in the words of the foreigner who had fallen into the water, " I will drown, nobody shall help me."

It has been said very truly that " Can't do mathematics " generally means " won't use paper and pencil." There is a mental inertia which makes it possible for students to sit and stare at a set of equations, and wonder how the result ever comes out, without once trying to see what would happen if they themselves carried out the processes suggested. Of course these are the very ones who will complain most bitterly of the amount of time they have spent on the assignment.

There are also the students who are willing to spend time, but are strangely reluctant to spend brain energy. With what a

sense of virtue the young student displays three pages of figures devoted to working an example for which a few lines are ample! Students who begin to see the charm of short cuts have often had their first introduction to the pleasure of mathematical thought.

It is easy to pile up charges against our unsatisfactory students; they are easily discouraged, they do not know how to apply themselves, they have no power of concentration, they have never learned the joy of hard work, they are too ready to ascribe their failures to causes outside themselves. (Note in this connection that in their vocabulary *flunk* is apt to be a transitive verb) they wish a painless education, they are bored by insistent demands for accuracy and regularity of work.

All this seems to indicate weakness of *spirit* rather than of mind, and it may seem to some of us that there is no cure comparable to a stiff course of mathematics. Certainly we may rejoice in the conviction that, even at the cost of some failures, many students have learned by this discipline lessons that have later made possible successes in many directions.

A recent report on business failures for the last three years shows that the number of failures each year is about 75 per cent. of the number of new firms starting in business. It may be a comfort at this point to consider how much more successful these young students are than business men. For, of the new students that come yearly to our schools and colleges, surely the percentage of failures never reaches 75. In point of fact the great majority do good work, while a very large majority pass in all their mathematical subjects.

And so we may be sure that, despite the difficulties inherent in mathematics, they are not too great for our students to cope with them successfully; despite the fact that all of us who are teaching mathematics are not doing it perfectly, we really are meeting with some success; despite the evil reports of some unwise and ignorant critics, their influence is not all-powerful; and despite the faults and follies of some few students, the large majority work with spirit and resolution.

For the sake of our students, and no less for the sake of our chosen subject, we must work to lessen the number of preventable failures, never by lowering our standards, but rather by

raising our ideals for teachers and students alike, so that all who teach mathematics shall be equipped with wisdom and patience and insight and enthusiasm, and shall be able not only to hold their students to regular, accurate, thorough work, but also to inspire them all along their course with the interest, the charm, and the far-reaching possibilities of this study.

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A SOLUTION OF EQUATIONS BY STANDARD CURVES.

By R. C. Colwell.

If the general quadratic is expressed in the form

$$x^2 - kx + l = 0$$

$a + b = k$ can be represented by a straight line which cuts the
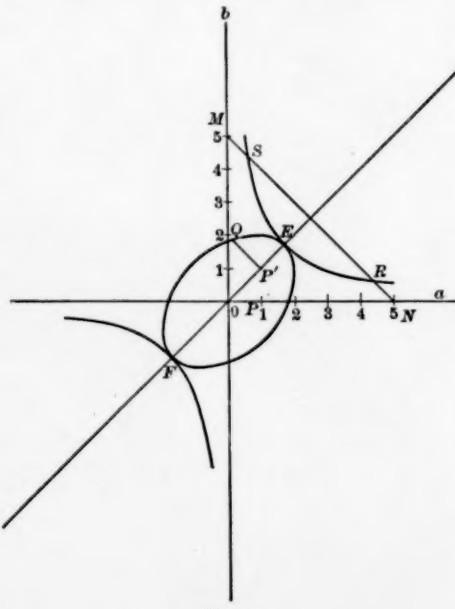
Solution: $x^2 - 5x + 3 = 0$, at R, S.
Solution: $x^2 - 2x + 3 = 0$, $OP + P'Qi$.

hyperbola $ab = l$ in two points the co-ordinates of either of which are a and b, the roots of the quadratic.

Example (see Fig. 1):

If $x^2 - 5x + 3 = 0$,

The line $a + b = 5$ cuts the hyperbola $ab = 3$ in the point R, S giving the solution $x = 4.3$ or $.7$.

As the value of $k^2 - 4l$ becomes smaller, l remaining constant, the line MN slides toward the origin, becoming tangent to the hyperbola at $k^2 - 4l = 0$, and not cutting the hyperbola at all when $k^2 - 4l < 0$. The roots of the equation are then complex and are determined by a point on the ellipse whose major axis is the major axis of the hyperbola and whose semi-minor axis equals $5l$. The real part of the complex root is evidently $k/2$, which is measured from the origin along the x axis. At the point thus found, a perpendicular is erected to cut the major
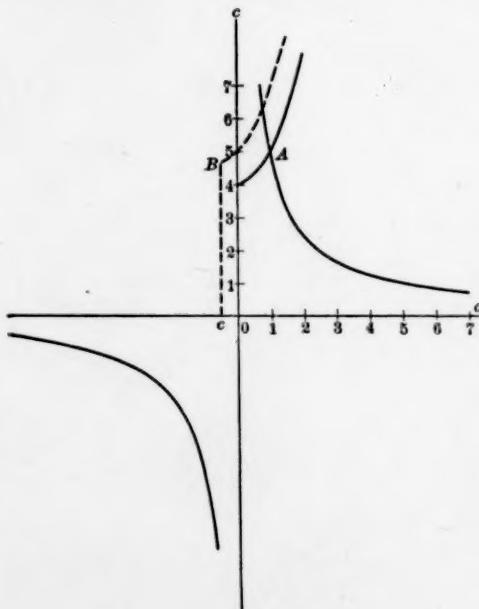

FIG. 2.

Solution: $x^3 + 4x - 5 = 0$, real root at A.

Imaginary roots, $OC \pm i \sqrt{BC}$.

axis of the ellipse. The perpendicular to the major axis from the point thus obtained cuts the ellipse and its length is the coefficient of i in the imaginary part of the root.

Example (Fig. 1):

If $x^2 - 2x + 3 = 0$,

The real part of the root is $OP = 1$;

The imaginary part is $P'Qi = 1.4i$;

The complete root $x = 1 \pm 1.4i$.

In the reduced cubic equation

$$x^3 + px + q = 0.$$

If a is one of the roots and c the product of the other two, then $a^2 - c = -p$ and $ac = -q$. Now $a^2 = c = -p$ is a parabola and $ac = -q$ is an hyperbola, and their points of intersec-
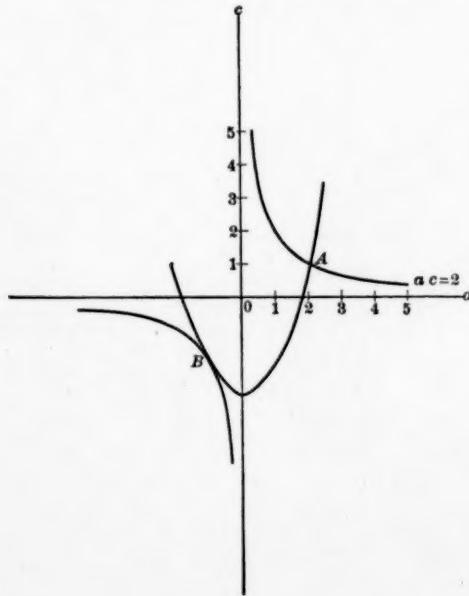

FIG. 3.

Solution: $x^3 - 3x - 2 = 0$. Roots at A and B.

tion will give the values of a and c, and therefore the roots.

Example (Fig. 2):

To solve $x^3 - 3x - 2 = 0$.

$a^2 - c = 3$. Therefore the vertex of the parabola $a^2 - c$ is at -3 on the y axis. It then cuts the hyperbola $ac = 2$ in the

point whose abscissa is 2 and touches the curve at the point whose abscissa is -1. Therefore $a=2$, $c=1$, or $a=-1$, $c=-2$, from which the solution 2, -1, -1 follows at once. If there is only one real root the parabola will cut the hyperbola in only one point; but the quadratic of the other two roots can be written at once and solved by the methods given before, or by the following method.

Example: $x^3+4x-5=0$.   (See Fig. 3.)

The parabola $a^2-c=-4$ cuts the hyperbola $ac=5$ in the point 1, 5.   Therefore the real root is 1.

The quadratic of the remaining roots is

$$x^2+x+5=0.$$

This is solved by moving the major axis of the parabola, $a^2=c$, which has been cut out, to the line $x=-\frac{1}{2}$ and letting it cut the y axis at 5.   The ordinate of the vertex is the square of the coefficient of i.

The solution is then $-.5\pm i\sqrt{4.7}=-.5\pm2.18i$.

The solution of a cubic in the reduced form

$$x^3+(c-a^2)x-ac=0$$

depends upon ac, and a^2.   The product ac may be found on the D scale of a slide rule with a and a^2 on the C and B scales respectively.   It is therefore possible to solve a cubic in the above form upon a slide rule.*

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   * Runge, "Graphical Methods."

WAR PROBLEMS IN MATHEMATICS.

By William E. Breckenridge.

In the preparation of this paper the author was assisted by Murray K. Aronson, Mary Baxter, R. P. Bliss, Morris Cohen, Mary Fowler, Earnest Greenwood, Marie L. Guinée, Lena M. Keeler, H. F. MacNeish, L. S. Odell, Anna M. Ohlssen, A. Lila Pratt, Jeannette Romaine, Earl S. Russell, Frank C. Touton, Earnest Townsend, Agnes L. Waring, students in Teachers College.

The drawings were made by M. C. Henriques.

This collection of problems has for its purpose the teaching of patriotism in the mathematics classroom. It aims to suggest to teachers of mathematics in colleges, high schools, junior high schools, and the upper grades of elementary schools types of war problems that may be used in the mathematics class either as substitutes for the old familiar problems or as supplementary illustrative material. In the discussion of these problems there will be presented excellent opportunities for the teacher to interest his students in some of the splendid efforts of our coun-


Fig. 1.
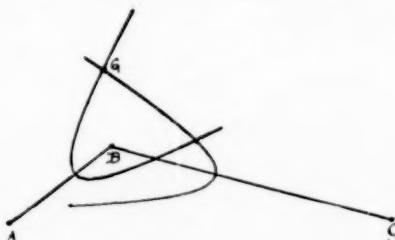
try and our allies to win the war, thereby inspiring these students with zeal for patriotic service.

The material is suggestive rather than exhaustive. Many good problems that were contributed have been omitted on ac-

count of lack of space.   Special mention should be made of Mr.
H. F. MacNeish, whose experience in the Yale R. O. T. C.
enabled him to contribute many valuable problems.

1. *Locating the German Super-Gun.* (The following method
was actually used.)   *A, B* and *C* are three observation stations
equipped with chronometers registering time accurately to one
tenth of a second.   The distance $AB = 1,000$ ft., $BC = 3,000$
ft.   Station *A* records the report of the gun 0.6 second later
than *B*.   How much nearer to *B* than to *A* is the gun located?
What is the locus of all possible positions of the gun?   (One
branch of a hyperbola adjacent to *B* with foci at *A* and *B*.)
Station *C* records the report of the same gun 2.4 seconds later
than station *B*.   How much nearer to *B* than to *C* is the gun?
What is the locus of all possible positions of the gun?   (One
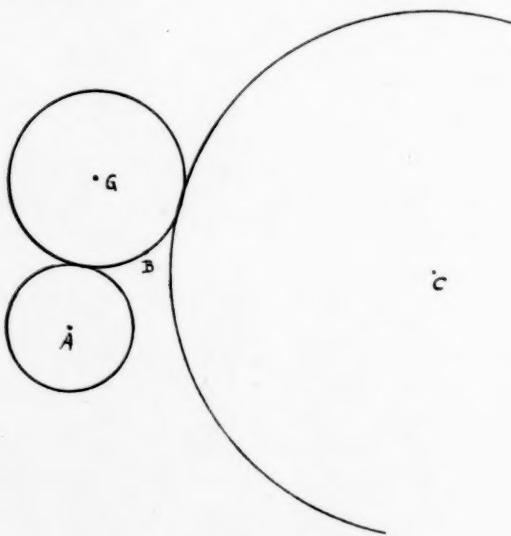

Fig. 2.

branch of a hyperbola adjacent to *B* with foci at *B* and *C*.)
Locate the gun graphically at the intersection of these two foci.

2. The problem of locating the super-gun may be solved by
elementary geometry.

Assume that sound travels 1,000 ft. per second.   Taking as large a scale as possible, draw a circle with A as a center and a radius of 600 ft.   With C as a center and a radius of 2,400 ft., draw another circle.   Through B draw the circumference of a circle tangent to circles A and C.   The center of this circle will be the position of the super-gun.

It is possible to draw this circle by elementary geometry, but a more practical method is to locate the center of the circle by trial, using a very large scale.

3. The same problem may be stated more simply as follows:

Three listening stations, A, B and C, are established near the front line. B is equipped with a listening device and an electric connection with stop watches located at A and C.   When the sound of the gun is heard at B, the observer presses an electric button, starting stop watches at A and C.   When the observers at A and C hear the report of the gun, they observe the time that has elapsed since their watches were started.   Suppose 2
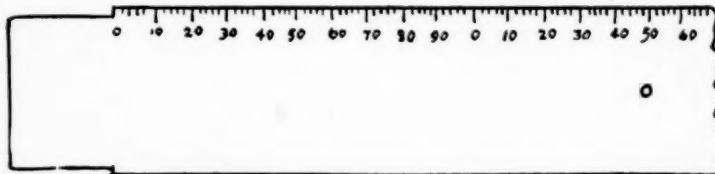
FIG. 3.
Battery Commander's Ruler, Left Half, exact size.

seconds have elapsed at A and 3 seconds at C.   Sound travels approximately 1,000 ft. per second, but the velocity could be determined as accurately as possible under the conditions at the time of the observations.   Then A is 2,000 ft. farther from the gun than B, and C is 3,000 ft. farther from the gun than B. Draw a circle with A as a center and 2,000 ft. as a radius. Draw another circle with C as a center and 3,000 ft. as a radius. Draw a circle whose circumference shall pass through B and which shall be tangent to the circles A and C.   The center of the circle is the required location of the super-gun.

The B. C. Ruler is a ruler about 6 in. long with a hole at the

middle graduation through which runs a string long enough to reach from the ruler to the observer's teeth when the ruler is held at arm's length. As used at the Yale R. O. T. C., the ruler is calibrated so that when the knot on one end of the string is held in the teeth, the six inches of the scale subtends 300 mils. The string is then about 20 in. from the ruler and the graduations on the ruler are numbered as indicated in Fig. 3. To
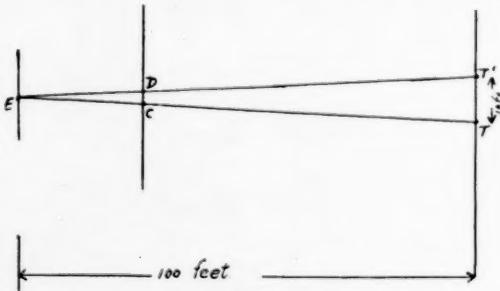
FIG. 4.

check the calibration of the ruler, a convenient distance of perhaps 100 ft., is measured and a rod 10 ft. high as TT' is noted. The observer holds the ruler vertically so that the line of sight from his eye, E, to T, the lower end of the object, intersects the ruler at its midpoint. He then notes the point D on the ruler where his line of sight to T' intersects the ruler. The distance CD on the ruler is called 100 mils since it subtends an angle of 100 mils. One hundredth of CD is therefore 1 mil. A mil is the angle whose tangent is 0.001, *i. e.*, the angle subtended by 1 ft. at a distance of 1,000 ft. It is approximately $\frac{1}{6400}$ of 360°.

A simple mil rule may be made as follows: To one end of any 6-in strip of wood or a lead pencil attach a string 20 in. long with a knot at the end of the string. Mark off the strip of wood into inches. Every inch will be 50 mils. Subdivide the inch into as small divisions as are practical. Transparent celluloid may be used to make a very good ruler.

If $R =$ the range in yards,

$\qquad W =$ the width or height of an object in yards at the distance of the range,

$\qquad M =$ the number of mils subtended by W,

then, evidently, $R = \dfrac{W}{M} \times 1,000.$

4. Prove this formula by similar triangles.

Estimation of Range by the B. C. Ruler.

5. A certain tree is estimated to be 15 yd. high. It covers an angle of 25 mils on the B. C. Ruler. How far away is it?

$$R = \frac{W}{M} \times 1{,}000$$

$$= \tfrac{15}{25} \times 1{,}000$$

$$= 600.$$

6. The telegraph poles seen on a distant railway running at right angles to our line of sight are known to be 44 yd. apart. The distance between two adjacent poles subtended 40 mils on the B. C. Ruler. Find the range to the railway.

$$R = \tfrac{44}{40} \times 1{,}000$$

$$= 1{,}100.$$

7. A line of skirmishers at about 1 yd. per man apart covers 40 mils on the B. C. Ruler. The range is known to be 800 yd. Find the number of men.

$$W = \frac{RM}{1{,}000}$$

$$= \frac{800 \times 40}{1{,}000}$$

$$= 32.$$

Hence there are 33 men, one more than the number of spaces.

8. A column of infantry in fours is seen by a patrol at 1,200 yd. range. It is moving across he patrol's front and covers 120 mils on the B. C. Ruler from the head to the rear of the column. Assuming that the men are 1 yd. apart, how many men are in the column?

$$W = \frac{RM}{1{,}000}$$

$$= \frac{1{,}200 \times 120}{1{,}000}$$

$$= 144.$$

Hence the number of men in a line is 145 and in four lines $4 \times 145 = 580$.

9. From a certain point a section of a hostile trench measures 150 mils. A scout goes forward 330 paces and finds that the trench covers 200 mils. If a pace is counted as 1 yd., find the range from the first point to the trench.

$$R = \frac{W}{M} 1,000$$

and

$$R' = \frac{W}{M'} 1,000.$$

Hence

$$\frac{R}{R'} = \frac{M'}{M}.$$

By division

$$\frac{R}{R - R'} = \frac{M'}{M' - M}.$$

Hence

$$R = \frac{M'(R - R')}{M' - M}.$$

Applying this formula,

$$R = \frac{200 \times 330}{200 - 150}$$

$$= 1,320.$$

10. When the distance is paced away from the trench, find the formula for R, and make up problems under this formula.

11. We have come to a river bank, and a village at some distance across the river covers 150 mils. An observer walks back 200 yd., keeping our party on the bank in line with the village, and finds that at that point the village covers 120 mils. Find the distance from our position on the river bank to the village.

12. From the New Jersey shore of the Hudson River at Fort Lee Ferry, 4 freight cars on the New York shore subtend an angle of 40 mils. If a freight car is approximately 40 ft. long, find the width of the Hudson River at this point.

13. An artillery officer wishes to station a gun on a north-and-south road so as to fire to best advantage on a column of infantry 600 yd. long stationed on an east-and-west road. The front of the column is 200 yd. from the intersection of the two roads. Show that when CG is tangent to the circle through GAB, the angle AGB is greater than any other angle such as
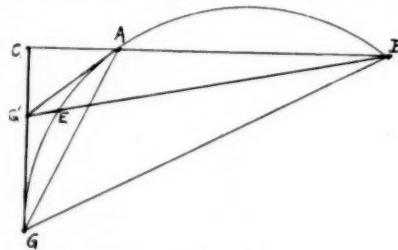
FIG. 5.

$AG'B$, since the angle AGB is measured by ½ arc AB and the angle $AG'B$ is measured by $\frac{1}{2}(AB - AE)$.

14. Also
$$\overline{CG}^2 = CA \times CB,$$
$$= 200 \times 800,$$
$$= 160{,}000,$$

and
$$CG = 400.$$
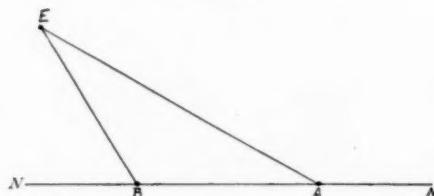
FIG. 6.

15. A battery is advancing along a road MN while attacking the enemy at position E. At position A, the angle EAN is measured with the B. C. Ruler. The battery continues to advance to the position B where the angle $EBN = 2$ angle EAN. Show that the range BE is equal to the known distance AB.

16. Two batteries of artillery situated 600 yd. apart in a second-line trench move their guns forward 100 yd. and 200

yd. respectively. Locate a common ammunition dump in the second-line trench to the most advantage with respect to distance. Producing $B'A'$ its own length to E and drawing BE intersecting AA' in D, we have D, the point required. Show

$$A'E = A'B'$$
$$AA' = 600$$
$$AB = 200$$
$$A'B' = 100$$
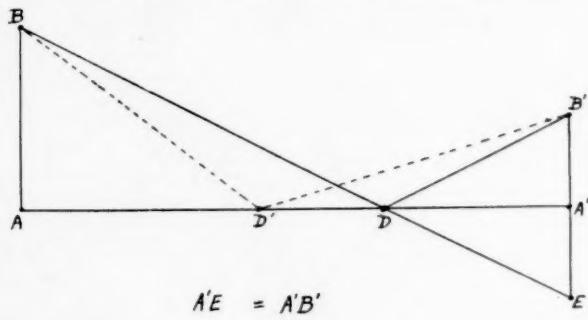
FIG. 7.

that $BD + DB' <$ any other distance as $BD' + D'B'$. Show that the numerical value of AD is 400 yd.

17. *Making Range Tables for Big Guns.*—For a brief, but
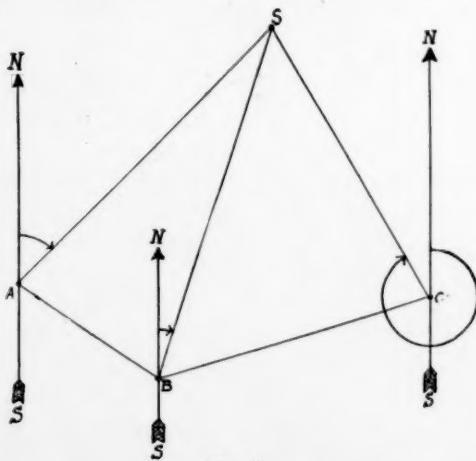
FIG. 8.

excellent description of the mathematics of artillery, see the article by Mr. J. Malcolm Bird on " The Mathematics of War-fare" in THE MATHEMATICS TEACHER, September, 1917. At the Sandy Hook Proving Grounds and at Aberdeen the range tables are corrected by actual results observed when a gun is fired. Computations involve the use of trigonometry and the slide rule. The text-book used by the Yale R. O. T. C. con-tains much material for problems in mathematics. Send to the Yale University Press, New Haven, Conn., for Notes on Train-ing Field Artillery Details by Danford and Moretti. The B. C. Ruler may be obtained at the same address.

Observation towers are located at A, B and C in the vicinity
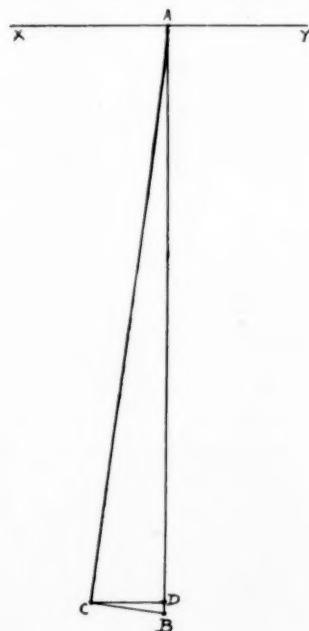
Fig. 9.

of the emplacement of the gun to be tested. NS represents a north-and-south line. S is the position of the splash made by the shell from the gun. AB, AC and BC are known distances. When the splash occurs, the observers at A, B and C take the angles NAS, NBS and NCS, measured clockwise and called the

azimuth of the splash. Three stations are used in case one observer fails to get the angle and as a check on the other observations. Problems based on the above figure are easily made up by assuming angles and distances.

Range Finding by Geometry.   (Used by scouts in the U. S. Army.)   See Fig. 9.

18. Scouts on the line CD see the enemy on the line XY. One man stands at D, another walks back along line DB, while a third walks to the left at right angles to AB for a convenient distance as CD.   By means of the optical square (or a forester's mirror) the right angle ACB is observed and the observer walking along DB is signalled to stop at. B. CD and DB are measured.   Hence,

$$AD = \frac{\overline{CD}^2}{DB}.$$

If CD is 80 yd. and DB is 10 yd. show that $AD = 640$ yd.

19. *Visibility of the Periscope of a Submarine.*—Show that the distance in miles at which a periscope may be seen from an observation station on a ship h ft. high is approximately $\sqrt{3h/2}$.

20. What must be the altitude above sea level in order that the periscope of a submarine (considered as at the surface of the water) may be visible 8 mi. away.   (Assume the diameter of the earth as 8,000 mi.)

21. At a distance of 50 ft. above sea level, at what distance is a periscope visible, if it extends 6 ft. above the surface of the water?

22. *Laying Out a War Garden.*—Fig. 10 will suggest an instrument for laying out a right angle in a war garden.   Why is the angle C a right angle?   What other dimensions could be used?

23. A camp site is roughly in the shape of a triangle whose sids are 1½ mi., 1¾ mi. and 1¼ mi.   Find the number of acres included in the tract.

24. An army camp site is bounded by two roads parallel to each other and 1¼ mi. apart.   The camp extends along one of them 1 mi. and along the other 1¾ mi.   The other two sides are straight roads 1⅜ mi. and 1⅝ mi.   Find the area of the

camp and the length of a road parallel to the first two and midway between them.

25. The floor area of the barracks erected at a camp was found to be too small; so it was decided to construct a new building
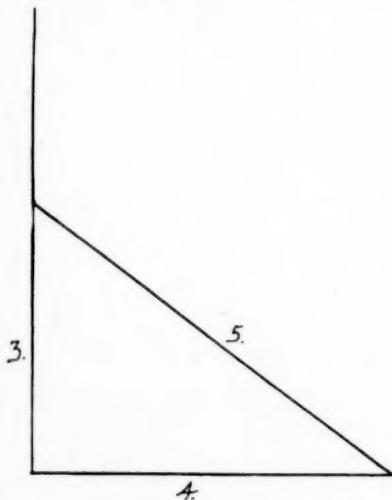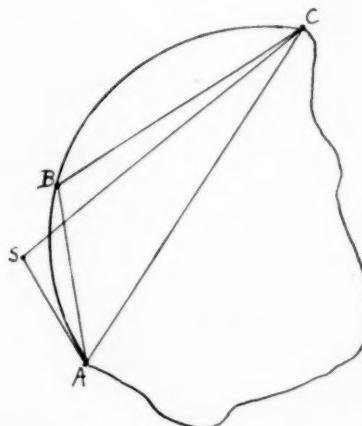
FIG. 10.   FIG. 11.

similar in shape but having a floor area 50 per cent. greater. If the first building was 60 ft. by 25 ft., find the dimensions of the second.

26. It is known that a certain harbor and entrance have been mined within a section determined by the circular arc shown. Show that a ship is safe as long as the angle S is less than the angle ABC. (Fig. 11.)

27. Represent graphically the weekly contributions of a school War Savings Society listed below:

Feb. 11, 1918..............$11.88   Feb. 18, 1918..............$14.53
 "   25, " .... 15.40   Mar. 4,   " 11.29
Mar. 11, " .... 14.13   "   18, " 11.04
 "   25, " .... 20.55   Apr. 1,   " ....Vacation
Apr. 8, " .... 22.15   "   15   " $14.18

28. *War Substitutes.*—Make a graph showing the percentage composition of foods given in the table below, representing the

proteid by a dotted line and the carbohydrates by a continuous line. Approximately how many pounds of potatoes do we need to substitute for one pound of wheat?

Is there any loss of carbohydrates by substituting 1 lb. of corn meal for 1 lb. of wheat?

	Per Cent. Proteid.	Per Cent. Carbohydrates.
Flour (white)	11.4	75.1
Flour (entire)	13.8	71.9
Corn meal	9.2	75.4
Rice	7.8	79.0
Potatoes	2.2	18.4
Oatmeal	16.1	67.5
Rye flour	6.8	78.7
Barley meal and flour..............	10.5	72.8

29. An army cook wishes to prepare a meal that will contain 8 per cent fat. He is going to use beef containing 6 per cent fat per pound and beans 3 per cent fat per pound. If he must prepare 50 lb. of food, how many pounds of beef must he use and how many pounds of beans?

30. The government desires to buy wheat from a certain dealer. This man shows two different samples of spring wheat, selling at $2.05 and $2.25 per bushel respectively. If the government then decides to buy 12,000,000 bushels at $2.10 a bushel, to be made up of the two kinds, how many bushels of each kind must be taken?

31. In preparing the soil for corn, a fertilizer containing 11 parts phosphoric acid, 12 parts nitrogen and 2 parts potash is used. How many pounds of each will be needed to make a ton of fertilizer?

32. A messenger with sealed orders is sent forward to a station 250 mi. distant. He travels at the rate of 20 mi. per hour. Thirty minutes later the orders are changed, and a second man starts out who travels at the rate of 35 mi. per hour. Where and when will be overtake the first man?

33. One girl scout sold 100 Thrift Stamps in three days, another girl scout sold the same number of stamps in four days. At the same rate how long will it take them both to sell 100 stamps?

34. In a certain meeting of the local Chapter of the Red Cross

with 48 members present, a motion was carried by a majority of 18. How many voted for the motion and how many against it?

35. The new American destroyers will make 35 knots an hour. In what time will a new destroyer overhaul a German submarine sighted at 3 mi. distant, if the submarine can travel only 15 knots on the surface?

36. The time of fall of a bomb, when the height at which it is dropped is known, is given by the following formula:

$$T = \frac{\sqrt{H}}{4} + \frac{H}{9,000},$$

where T is the time in seconds and H is the height in feet.

If a machine is flying at the height of 9,000 ft., what will be the time of the fall.

37. The distance a bomb travels forward:

$$D = \left[\left(\frac{\sqrt{H}}{4} + \frac{H}{9,000} \right) \times \frac{\text{Speed of machine}}{\text{in feet per second}} - \frac{H}{40} \right]$$

$D =$ distance in feet,

$H =$ height in feet.

A machine is at a height of 18,000 ft., and its speed is 100 ft. per sec. Find the distance the bomb travels forward from the moment of release to the time of striking the ground.

38. At what height is a machine from which a bomb dropped to the ground will travel forward a distance of 2,500 ft., the speed of the machine being 110 ft. per sec.?

39. At 40 cts. a day for every soldier how long could the 700 million dollars which the United States people waste in food products every year supply a regiment of 2,000 soldiers? Give the answer in years and months.

40. If the box of an army truck is 6 ft. long. 5 ft. wide and 2 ft. high, how many cubic yards of sand would it hold when completely filled?

41. What is the longest stick that would just go in from one corner to an opposite one of the above box?

42. An army truck runs about 8 mi. to a gallon of gasoline. If gasoline costs 25 cts. a gallon, how much would it cost for a trip of 116 mi.?

43. If a mess sergeant is allowed 50 cts. a day for every soldier in his mess and he spends only 46 cts. a day for each of 52 men for 6 days, how much extra would he have to spend on a Sunday dinner for the whole mess?

44. A train of 30 army trucks runs from Buffalo to New York, a distance of about 447 mi. Suppose they make 90 mi. a day. The gasoline consumption averages 9 mi. per gallon at 25 cts. a gallon, and the oil averages a gallon at 50 cts. for each 120 mi. Two soldiers ride on every truck and they are allowed $2.50 each for every day they are on the trip. How much would the whole trip cost?

45. If wool for knitting socks costs $3.25 a pound and it takes 5 oz. for a pair of socks, how much will the wool cost for 6 pairs of socks?

46. Find the difference in income for 10 yrs. from $500 invested in a savings bank paying 4 per cent. interest and the same amount invested in bonds of the Third Liberty Loan at $4\frac{1}{4}$ per cent. interest. In each case the interest payments are put in the bank every 6 mo. and draw interest at 4 per cent.

47. A certain insurance company offered to sell $1,000 bonds of the Third Liberty Loan bearing interest at $4\frac{1}{4}$ per cent. for 10 semi-annual payments of $100 each and in the meantime insure the buyer for $1,000. If the company kept all of the interest coupons, what rate of interest did the company get on its investment?

48. How much money invested in U. S. bonds of the Third Liberty Loan at $4\frac{1}{4}$ per cent. will give the same return as $850 invested in the first loan at $3\frac{1}{2}$ per cent.?

49. Two thousand boys left the New York City high schools to work on farms last summer for five months. They received $3.75 a week for the first four weeks, $5 a week for the next four weeks and $6.25 a week for the next four weeks, etc. Find every boy's weekly wage at the end of the summer, his total receipts for the summer, and the amount earned by the 2,000 boys during the five months.

50. The French army is using at the present time a type of trench digger that cuts a trench 10 ft. deep, 3 ft. wide and 500 ft. long in an hour. Assuming that it takes the same time for a trench 20 ft. deep but half as long, how long a time will it take to dig a 20-ft. trench 2,000 ft. long?

51. How many machines would have to be used to dig two 10-ft. trenches on a 10-mile front in one night of nine hours?

52. How much material in cubic yards is removed in going 16,000 ft.?

53. What is the height of the mound of earth thrown up if it spreads over only 6 ft. of ground and takes the mean position of a triangular cross-section? If it takes the form of a semi-circle? Banked up to form a quadrant?

54. How long must the trench periscope be for a trench 20 ft. deep with earth thrown out so that the cross-section is an equilateral triangle with a base of 8 ft.?

55. A man buys "Ex steamer" 1,000 bbls. of currants for $44,500. This includes 1 per cent. commission, 8 per cent. marine and war insurance, and 110 shillings per ton freight. Besides this he pays ½ per cent duty, 45 cts. on a barrel storage per month and 1 per cent storage insurance. (a) For what must he sell the currants per bbl. to the retailer to make 33⅓ per cent profit net? (steamer's tonnage = 2,240 bbls.). (b) If a bbl. weighs 300 lb., for what must the retailer sell the goods per pound to make the same net profit?

56. It is estimated that it requires one workman per ton per month to build a ship. How many workmen were engaged in 1917 to build the output of 800,000 tons?

57. How many men will be required to build 5 million tons of shipping per year called for by the Shipping Board if the men speed up 10 per cent?

58. If a factory turns out 250 guns a day and 2 tanks a week, how long will it take to turn out 50 tanks, at the same time making guns, providing the force is increased 2½ times?

59. How many boats will it take to make a pontoon bridge over a river 295 ft. wide if 5 boats are used every 25 ft.? In what time could the bridge be erected if it takes three minutes to fix two boats?

60. What is the cost per hour of a 16-candle power electric light when by the daylight saving law, 8 lights burning on an average of three instead of four hours every evening, there is a saving of 28 cts. per week?

61. *Proteid Values.*

$$
\begin{array}{l}
\text{1 pound cheese} = 1.27 \text{ pounds sirloin.} \\
\quad (\text{cottage}) \quad = 1.09 \text{ round steak.} \\
\qquad\qquad\quad = 1.52 \text{ chicken.} \\
\qquad\qquad\quad = 1.44 \text{ smoked ham.} \\
\qquad\qquad\quad = 1.31 \text{ leg of lamb.} \\
\qquad\qquad\quad = 1.37 \text{ breast of veal.}
\end{array}
$$

In your section of the country find the cost of the above articles of food.

Find the cost of the meat equivalent to one pound of cheese.
How much is saved in using cheese and what per cent.?

62. The volume of a submarine boat is 4.5 cubic meters and the weight of the hull and its contents is 4,150 kg. What volume of water will have to be taken into the boat to sink it?

63. When the Germans invaded Belgium the Belgians managed to run 1,900 engines over the border into France. Of these 800 were rusted and beyond repair. The rest were repaired and turned over to the army for service. What per cent were serviceable?

64. Find how many ounces of pork and how many ounces of bread would be necessary for a standard ration (4 oz. fat. and 4 oz. proteid) if pork is 25 per cent fat and 12 per cent protein and bread is 1 per cent fat and 9 per cent protein?

Many such problems can be made up from any food table.

65. If whenever you cast on 72 stitches for a sweater, you bind off 18 stitches for the neck, how many would you bind off if you had used smaller needles and cast on 84 stitches?

66. Two airplanes starting at the same time from the same aviation field fly in direct lines that form an angle of 30°; one flies at 35 mi. an hour, the other at 70 mi. an hour. How far apart will they be at the end of 15 min.?

67. Two towns, A and B, are 70 mi. apart and are connected by a road. A French army is situated in a moor 80 mi. from A and 55 mi. from B. Find the shortest direct distance that this army can take to reach the road AB.

68. A division marches 12 mi., turns through 60° to its left, goes 4 mi., turns through 90° to its left and marches ½ mi. farther. How far is it from the starting point?

69. Two roads branch at A. They make an angle of 70°.

One division marches 6 mi. from *A* along one road, another marches 8 mi. from *A* along the other road. How far are the two divisions from each other?

70. From two points, *A* and *B,* on a line of earthworks, the bearings of a German Battery are respectively N. 16° E. and due north. If *B* is east-southeast of *A* and 2,000 yd. from *A,* find the distance of the gun from *A.*

71. Find the amount of $412 in five years' interest compounded quarterly a 4 per cent per annum.

72. Find the amount which should be paid for a $5 note due in 5 years, interest compounded quarterly at 4 per cent per annum.

73. Determine the dimensions of a cylindric can for canned vegetables so that the amount of tin used shall be a minimum and the can shall hold a quart.

74. Determine the minimum amount of seams to be soldered for the above can, assuming that the two circular edges and one lateral seam are soldered.

75. Combine the two and determine the most economical can, assuming cost of tin and solder. The time element of labor might be included.

76. Determine the minimum dimensions for a rectangular parallelopiped container that shall have a volume of 150 cu. in.

77. Determine the ratio of the length to the diameter of the cans so that they can be most economically packed in a given packing case.

78. How much railroad equipment is needed to move an army?

A statement prepared by Lieut. Col. Chauncey B. Baker, of the Quartermaster corps, U. S. A., and distributed to the railroads of the country by the Special Committee on National Defense of the American Railway Association is as follows: To move one field army of 80,000 men, consisting of three infantry divisions, one cavalry division, and a brigade, technically known as a brigade of field army troops—troops auxiliary to the infantry and cavalry divisions—requires a total of 6,229 cars made up into 366 trains with as many locomotives. These 6,229 cars would be made up of 2,115 passenger cars, 385 baggage, 1,055 box, 1,899 stock, and 775 flat cars. This quantity of equipment represents .7 of 1 per cent of the locomotives

owned by American railroads, 4.2 per cent of their passenger cars and 0.2 of 1 per cent of their freight equipment.

The railroad equipment required to move various organizations of the army at war strength is as follows:

Infantry regiment, 55 officers, 1,890 men, 177 animals, 22 vehicles. Cars required, 48 passenger, 5 baggage, 15 box, 9 stock, 8 flat; total, 85 cars.

Cavalry regiment, 54 officers, 1,284 men, 1,436 animals, 26 vehicles. Cars required, 36 passenger, 8 baggage, 25 box, 72 stock, 9 open; total, 150 cars.

Artillery regiment, light, 45 officers, 1,170 men, 1,157 animals, 32 vehicles, 24 guns. Cars required, 32 passenger, 9 baggage, 25 box, 58 stock, 46 flat; total, 170 cars.

Artillery regiment, horse, 45 officers, 1,173 men, 1,571 animals, 35 vehicles, 24 guns. Cars required, 34 passenger, 10 baggage, 25 box, 78 stock, 47 flat; total, 194 cars.

Artillery regiment, mounted, 45 officers, 1,150 men, 1,229 animals, 24 guns. Cars required, 30 passengers, 7 baggage, 30 box, 61 stock; total, 124 cars.

Engineers. Pioneer Battalion, 16 officers, 502 men, 165 animals, 12 vehicles. Cars required, 14 passenger, 2 baggage, 10 box, 8 stock, 4 flat; total, 38 cars.

Signal Corps. Field Battalion, 9 officers, 171 men, 206 animals, 15 vehicles. Cars required, 6 passenger, 2 baggage, 5 box, 10 stock, 5 flat; total, 28 cars.

Any number of questions may be based upon these facts, *e. g.*, find the equipment owned by American railroads, remembering that passenger and baggage cars are classed as passenger cars; box, stock and flat cars as freight cars.

79. How Resources Indicate Victory. Graph Work.

COAL AND IRON. PERCENTAGE IN DIFFERENT COUNTRIES.
PRODUCTION, 1913.

	Percentages.		
	Coal.	Iron.	Steel.
United States.....................................	38	40	42
British Empire....................................	26	14	12
France, Belgium, Italy............................	4	10	11
Germany, Austria-Hungary	25	27	28
All other countries	7	9	7

RESOURCES, 1913.

	Percentages.	
	Coal.	Iron Ore.
United States	52	63
British Empire	23	18
France, Belgium, Italy	1	2
Germany, Austria-Hungary	6	2
All other countries	18	15

PERCENTAGE OF OTHER RESOURCES, 1913.

	Petroleum.	Cotton.	Copper.	Timber.	Gold.	Wealth.	Railroads.	Ships.
United States......	64	61	56	56	19	29	39	17
British Empire.....	2	24	11	4	63	18	18	42
France, Belgium, Italy...........				3		15	7	8
Germany, Austria-Hungary	2		3	11		19	10	11
All other countries..	32	15	30	26	18	19	26	22

TEACHERS COLLEGE,
    COLUMBIA UNIVERSITY,
      NEW YORK CITY.

ARITHMETICAL ERRORS MADE BY HIGH SCHOOL PUPILS.

By J. H. MINNICK.

During the school year 1915–16 a review arithmetic test was given to the entering pupils of the William Penn High School of Philadelphia. The questions were prepared by Dr. J. T. Rorer, head of the Department of Mathematics, William Penn High School, and Dr. George W. Flounders, examiner of the Board of Public Education of Philadelphia. An exact copy of the test is given below. The exercises were all printed on one sheet and there was ample space left after each exercise for the solution.

Time: 30 minutes.   Name....................   Section......
                                        September 27, 1915.

REVIEW TESTS IN ARITHMETIC.

Do all the work on this sheet.

1. Add:

$ 2.34
1.68
2.75
.62
.89
5.48
3.26
.92
.77
7.15
3.50
$

2. Multiply:

97568
7098

3. Divide:

9) 270568

4. Divide (without decimals):

1908) 72563 (

5. From fifty-four and five-eighths subtract thirty-four and three-fourths; divide the remainder by two.

6. How many pieces of ribbon, 15½ inches long, may be cut from a 10 yard roll and what is the length of the remnant?

7. Extend the items of the following bill, total, and apply the discount.

80

Philadelphia, Sept. 27, 1915.

Mr. R. U. Goodpay.
    Bought of THE WILLIAM PENN MERCHANDISE COMPANY.

64 yards	Voile	@ $ .62½ per yd.		
48 "	Calico	@ .06¼ " "		
20 "	Lawn	@ .12½ " "		
2 doz.	Fasteners	@ .33⅓ each	_____
		Less 2% discount,	_____
		Amount due,	_____

The instructors of the Department of Mathematics marked the test papers on the basis of " right " or " wrong," without an examination of the specific errors made by the pupils. Later, Dr. Rorer asked me to examine the manuscripts with a view of determining the various types of errors made by the pupils, and the relative frequency of each type. Accordingly, I have carefully examined the manuscripts of a little more than 1,000 pupils. The results* are recorded below.

It should be said that statistics of such an examination can only be approximately correct. The pupil's thoughts are but partially recorded on her paper and in some cases the examiner can only infer the type of error that leads to a given incorrect answer. Furthermore, several errors frequently occur in the same problem, and it is easily possible that some of these have escaped the attention of the examiner. However, we believe that the data given below are fairly accurate.

As the tests were given to incoming pupils who had received little or no training in the William Penn High School, the results must not be taken as an indication of the efficiency of mathematical instruction in that school. They do, however, indicate more or less definitely the store of arithmetical knowledge and habits which the teachers had to use as a basis for their work with these pupils.

In most cases permitting mistakes in number combinations and in " carrying," it was impossible to determine to which type of error an incorrect answer was due. Hence we have not considered these two errors separately, but have classed them as

* Dr. Rorer's report of the test is given in a leaflet published by Dr. Flounders under date of October 25, 1916. A further report was made by Oliver M. Long in *Current Education*, October, 1916.

mistakes in addition or multiplication. It is worth noting, how-ever, that in most cases where it was possible to recognize a mistake in "carrying" it was due to inverting the digits of a number. Thus, in $7 \times 8 = 56$ the pupil might have the correct result in mind but record it as 65.

If an error occurred in all the papers fewer than five times, it has not been included in the tabulation. The only errors pos-sible in Exercise 1 were either in number combinations or in "carrying." These were relatively few, and, as noted above, it was impossible to distinguish between them. Hence the data for Exercise 1 have not been included in this discussion.

Table I gives the data for Exercise 2. Thus, errors in multi-plication combination occurred 603 times, which was 78 per cent. of all the errors in the solution of Exercise 2. The errors in multiplying by zero were of two types; namely, using zero as unity and leaving the zero out of consideration in arranging the partial products.

TABLE I.

NUMBER OF TIMES EACH TYPE OF ERROR OCCURRED IN THE SOLUTIONS OF EXERCISE 2.

Type of Error.	Number of Times Each Type Occurred.	Per Cent. of Total Number of Errors.
Multiplication combination	603	78.0
Addition combination	96	12.4
Failure to multiply by one digit of multiplier	23	3.0
Omission of digit in answer or par-tial product	22	2.8
Multiplying by zero	21	2.7
Failure to multiply part of multi-plicand by one digit of multiplier	8	1.0
Total	773	99.9

At first it may seem that failure to multiply by one digit of the multiplier and failure to multiply part of the multiplicand by one digit of the multiplier are the same errors carried to dif-ferent degrees. This, however, is not the case. In the first the child has overlooked a part of the multiplier, but has used all of the multiplicand; in the second she has neglected a part of the multiplicand while she has used all of the multiplier.

Table II shows that Exercise 3 did not give as much trouble as Exercise 2. The mistake most frequently made in connec-

TABLE II.

NUMBER OF TIMES EACH TYPE OF ERROR OCCURRED IN THE SOLUTIONS OF EXERCISE 3.

Type of Error.	Number of Times Each Type Occurred.	Per Cent. of Total Number of Errors.
Error due to zero	120	61.2
Error in division	44	22.4
Dropped fraction	13	6.6
Error in fraction	10	5.1
Error in subtraction	9	4.5
Total	196	99.8

tion with zero was the omission of a zero in the quotient. That is, the pupils got the same result as if they had divided 9 into 27,568. By an " error in division " is meant an error in finding how many times the 9 is contained in the *next* number of the dividend. Perhaps dropping the fraction ⅑ should not be counted a mistake, as pupils are taught to do so in many cases. The errors in fractions were usually the misplacing of the decimal point or the writing of ⅓ instead of ⅑. A surprisingly large number of pupils used long division, and it was in this connection that the errors were made in subtraction. Another point of interest is the fact that pupils did not notice that ⅑ is a better form for the fraction than .11⅑.

TABLE III.

NUMBER OF TIMES EACH TYPE OF ERROR OCCURRED IN THE SOLUTIONS OF EXERCISE 4.

Type of Error.	Number of Times Each Type Occurred.	Per Cent. of Total Number of Errors.
Error in division	84	28.6
Error in subtraction	81	27.6
Error in multiplication	57	19.5
Error in handling remainder......	23	7.8
Failure to find fraction	18	6.1
Error in copying	16	5.5
Complete confusion	9	3.0
Error in reducing fraction to lowest terms	5	1.7
Total	293	99.8

The data for exercise 4 are given in Table III. The mistakes tabulated under "error in division" are mistakes in determining how many times the divisor was contained in the partial

dividends. Of the 84 pupils making such errors, 41 had a digit in their quotients too small. Thirty-one of these 41 pupils noticed that the remainder contained the divisor again. Making this division they obtained two or more digits which belonged in the same place. Thus, in the answer $371^{59}/_{1908}$ the 7 and 1 belong in units place and the answer therefore should be $38^{59}/_{1908}$, which is correct. The remaining ten pupils did not notice that the remainder was larger than the divisor and obtained a result in the form $37^{1967}/_{1908}$.

Of the 23 pupils making errors in handling the remainder, fourteen divided the final remainder by the divisor before expressing it in fractional form. As this remainder was less than the divisor, the resulting digit of the quotient was zero, giving an answer such as $380^{59}/_{1908}$. The remaining 9 pupils failed to divide the last time, resulting in an improper fraction, thus $3^{15323}/_{1908}$. The failure to express the remainder in fractional form perhaps should not be counted an error, as many teachers permit the pupils to express their answer as 38 with a remainder of 59.

The work of 9 pupils was so poorly arranged that it was impossible to follow their thought. In some of these cases it was evident that the pupil had no clear idea of what she was trying to do.

Exercise 5 involves three steps—writing numbers, subtracting, and dividing by 2, and, as Table IV shows, most of the errors can be classified under these three heads. Many of the errors are due to carelessness, such as the failure to divide by 2, multiplying by 2, subtracting the wrong direction and adding instead of subtracting. Twenty-six pupils divided the integral part of the difference by 2, but failed to divide the fraction. This may be due to carelessness, or it may be due to the fact that the pupil had forgotten how to divide fractions by whole numbers. The 5 pupils who divided 2 by the difference, no doubt, had confused the rule for division of fractions and inverted the dividend instead of the divisor.

Dividing the integral part of the difference by 2 gave a remainder of 1. Firty-six pupils were unable to handle this remainder. The most common mistake was to unite 1 with $\frac{7}{8}$, getting the result $1\frac{7}{8}$ instead of the sum of 1 plus $\frac{7}{8}$.

TABLE IV.

NUMBER OF TIMES EACH TYPE OF ERROR OCCURRED IN THE SOLUTIONS OF
EXERCISE 5.

Type of Error.	Number of Times Each Type Occurred.	Per Cent. of Total Number of Errors.
Errors relative to division by 2:		
Failure to divide by 2.........	137	16.6
Error in division by 2.........	67	8.1
Error in handling remainder..	56	6.8
Failure to divide fraction by 2.	26	3.1
Multiplied by 2	20	2.4
Divided 2 by the difference....	5	0.6
Total	311	37.6
Errors relative to subtraction:		
Error in number combinations.	107	12.9
Subtracted fractions wrong direction	92	11.1
Error in "borrowing" 1 from 54 and adding it to ⅝.......	78	9.4
Adding instead of subtracting.	59	7.1
Error in reducing fraction to lowest terms	56	6.8
Error in reducing mixed number to fraction	14	1.7
Error in reducing fraction to mixed number	14	1.7
Total	420	50.7
Error in writing numbers	46	5.6
Error in writing decimals	30	3.6
Complete confusion	15	1.8
Error in copying numbers	5	0.6
Total	827	99.9

Clearly the pupil had confused this case with such problems as
256 divided by 2. Dividing 5 by 2 we get 2 and a remainder of
1. This 1 is 1 ten, and together with 6 units makes 16 units.
The pupil learns to do this mechanically, without any thought of
the reasons back of the operation. When she divides $9\frac{7}{8}$ by 2,
she gets first 4 and a remainder of 1. This 1 and the $\frac{7}{8}$ natu-
rally make $1\frac{7}{8}$, just as the 1 and 6 make 16.

Ninety-two pupils subtracted the fractions the wrong direc-
tion. This no doubt was suggested by the fact that the fraction
of the subtrahend is greater than that of the minuend. Seventy-
eight pupils had trouble in "borrowing" 1 from 54 and adding
it to ⅝. Either they did not subtract the 1 from 54, or in adding

it to ⅝ they got ¹⅝. Here again the error was due to confus-
ing it with "borrowing" in whole numbers. The errors in the
reduction of fractions were usually errors in the fundamental
operations.

The errors in decimals usually resulted from an attempt to
reduce the original fractions to decimals before performing the
subtraction. The solution of 15 pupils indicated that they were
completely confused.

Table V gives the data for exercise 6. In all, 1,112 errors

TABLE V.

NUMBER OF TIMES EACH TYPE OF ERROR OCCURRED IN THE SOLUTIONS OF
EXERCISE 6.

Type of Error.	Number of Times Each Type Occurred.		Per Cent. of Total Number of Errors.
Errors in finding remnant:			
Found fraction of piece	463		41.3
Error in reducing fraction of			
piece to inches	80		7.1
Did not know the meaning of			
remnant	35		3.1
Did not attempt to find rem-			
nant	11		1.0
Total		589	52.5
Error in method of finding number of pieces:			
Divided length of piece by en-			
tire length	35		3.1
Multiplied length of piece by			
entire length	26		2.3
Total		61	5.4
Error in operation:			
Decimals	124		11.1
Division	50		4.5
Subtraction	34		3.0
Multiplication	32		2.9
Total		240	21.5
Use of denominate numbers:			
Error in reducing from one			
denomination to another	83		7.4
Divided without reducing to			
same denomination	35		3.1
Total		118	10.5
Complete confusion		114	10.2
Total		1,122	100.1

are recorded in this table. Slightly more than 52 per cent. of these errors were made in finding the remnant. This includes only mistakes in the method of finding the remnant and in the interpretation of the meaning of the word remnant; it does not include mistakes in computation and decimals. Four hundred and sixty-three pupils found the fractional part of a piece left and called it the remnant. Eighty pupils who recognized the real meaning of this fraction made errors in reducing it to inches. Thirty-five pupils did not know the meaning of the word "remnant"; some called the entire piece of ribbon the remnant, while others took the word to mean one of the pieces $15\frac{1}{2}$ inches in length.

The next largest number of errors was made in the use of decimals. Seventy of these were miscellaneous errors in placing the decimal point, 6 were made in reducing common fractions to decimals and vice versa, and the remaining 48 were made in finding the remnant. The pupil divided 360 by 15.5, getting 23 with a remainder of 35 instead of 3.5. The cause of the error is clear. The pupil has learned in a mechanical way to write the fraction $\frac{35}{155}$ omitting the decimal from both numerator and denominator. Hence in this case she very naturally omitted the decimal from the remainder.

The most common mistake in the use of denominate numbers was the confusion of the number of inches and feet in a yard. Many pupils used a yard as equal to 12 inches, and in a few cases they used it as equal to 3 inches.

The work of 114 pupils indicated that they had no idea as to the method of solving the problem. It is probable that in many of these cases the pupil did not visualize the act of cutting the roll of ribbon into pieces $15\frac{1}{2}$ inches long, or if she did, she did not see the relation of the arithmetical operations to this act.

In tabulating the data for exercise 7 (see Table VI), it has been our purpose to show where the errors have occurred, rather than the specific kind of arithmetical errors found in the papers. By far the largest number of errors was made in extending the items of the bill. These were for the most part mistakes in number combinations or in the use of decimals. Fifty-five pupils showed that they did not know the meaning of the word *discount*. Dividing the total by 2, subtracting 2 or .02 from the

TABLE VI.

NUMBER OF TIMES EACH TYPE OF ERROR OCCURRED IN THE SOLUTIONS OF
EXERCISE 7.

Type of Error.	Number of Times Each Type Occurred.		Per Cent. of Total Number of Errors.
Error in Extensions:			
64 × .62½	138		18.1
48 × .06¼	123		16.1
2 × .33⅓	117		15.4
20 × .12½	70		9.2
Total		448	58.8
Error in Discount:			
Computation	147		19.3
Did not know meaning of discount	55		7.2
Failed to find discount........	37		4.9
Failed to subtract discount....	26		3.4
Total		265	34.8
Error in adding to final total...........	27		3.5
Error in subtracting discount..........	22		2.9
Total		762	100.0

total, subtracting half the total from the whole total, and adding
the discount instead of subtracting it, are the more common
errors resulting from the misconception of discount. In some
cases the pupil found the discount and called it the amount due.
It is also possible that the 37 pupils who failed to find the dis-
count and the 26 who did not subtract it from the total did not
understand the meaning of the word *discount*. Many pupils
who had correct answers did not enter the amounts correctly.
However, the teachers who originally marked the papers did
not take these errors into consideration, and for that reason we
have not tabulated them.

It should be emphasized that general conclusions concerning
the relative frequency with which errors occur or their relative
values cannot be drawn from the results of such a test as this.
We do not know that this test gives equal opportunity for all
arithmetical errors to occur. However, it is clear that any
teacher may easily devise a set of questions which will yield
valuable knowledge concerning her classes and give her a fairly
definite idea of the arithmetical background which the children
have and which she must use as a basis for her work. Also
we can draw certain conclusions relative to the teaching of

arithmetic. Pupils often have difficulty with those operations which should be reduced to habit. If an operation is to be used frequently, economy of time and accuracy demand that it shall become habitual. This requires not only drill at the time the habit is formed, but also frequent applications in order to keep the habit fixed. Errors in number combinations are due to such incorrect or incomplete habit formation. This is especially true when zero is involved. Most teachers assume that pupils know the combinations of zero with other numbers. This assumption is not true. Even college students frequently have trouble in such cases. It is just as necessary to drill on $0 \times 8 = 0$ as on $9 \times 8 = 72$. If, however, an operation has applications in slightly different forms, an intelligent understanding of it may prevent errors when it is applied in a new form. Thus, in Exercise 5 the pupils "borrowed" 1 from 54 and combined it with $\frac{5}{8}$, getting $1\frac{5}{8}$. In dealing with integral numbers they had learned in a mechanical way to "borrow" 1 from 6 in 267 and combine it with 7 so as to get 17. If the pupils had had an intelligent understanding of this operation, if they had known that they were reducing 1 ten to units and adding it to 7 units, and if they had understood that numbers must always be reduced to the same denomination before they can be added, it is possible that such an error would not have occurred so frequently. Errors in application of arithmetic are frequently due to the child's inability to read the problem. This is the case with exercises 6 and 7. Some pupils did not understand the meaning of *remnant* and of *discount*. This does not mean that these words should not be used, but before beginning her solution the pupil should be sure that she understands their meaning. Successful application of arithmetic requires an analysis of the situation to which the application is to be made. If in such problems as Exercise 6, pupils were taught first to see the various operations involved in the physical act described and then to identify the arithmetical operations with the corresponding physical operation, fewer pupils would be confused by such problems.

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SOME RELATIONS CONNECTING THE SUMS OF THE COAXIAL MINORS OF A CIRCULANT.

By W. H. Metzler.

In 1878 Dr. J. W. S. Glashier called attention* to the fact that the determinant of order n formed from subtracting x from the elements along the principal diagonal of a continuant has as factors:

$$- (x - s) \prod_{k=1}^{m} \{x^2 - (a_1 + a_2\theta^k + a_3\theta^{2k} + \cdots + a_n\theta^{(n-1)k})$$
$$\times (a_1 + a_2\theta^{n-k} + a_3\theta^{n-2k} + \cdots + a_n\theta^{n-(n-1)k})\}$$

for
$$n = 2m + 1,$$

and

$$(x - s)(x - s') \prod_{k=1}^{m-2} \{x^2 - (a_1 + a_2\theta^k + a_3\theta^{2k} + \cdots + a_n\theta^{(n-1)k})$$
$$\times (a_1 + a_2\theta^{n-k} + a_3\theta^{n-2k} + \cdots + a_n\theta^{n-(n-1)k})\}$$

for
$$n = 2m,$$

where

$$s = a_1 + a_2 + a_3 + \cdots$$
$$s' = a_1 - a_2 + a_3 - \cdots$$

and θ is an imaginary nth root of unity.

Thus for $n = 5$ we have

$$\begin{vmatrix} a_1 - x & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 - x & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5 - x & a_1 & a_2 \\ a_4 & a_5 & a_1 & a_2 - x & a_3 \\ a_5 & a_1 & a_2 & a_3 & a_4 - x \end{vmatrix}$$

$$= - \{x - (a_1 + a_2 + a_3 + a_4 + a_5)\}\{x^2 - (a_1 + a_2\theta + a_3\theta^2$$
$$+ a_4\theta^3 + a_5\theta^4)(a_1 + a_2\theta^4 + a_3\theta^3 + a_4\theta^2 + a_5\theta)\}$$
$$\times \{x^2 - (a_1 + a_2\theta^2 + a_3\theta^4 + a_4\theta + a_5\theta^3)(a_1 + a_2\theta^3$$
$$+ a_3\theta + a_4\theta^4 + a_5\theta^2)\}.$$

If we represent the imaginary factor of the circulant by α_1, α_2, α_3, α_4, the right-hand side may be written

$$(x - s)(x^2 - \alpha_1 \cdot \alpha_4)(x^2 - \alpha_2 \cdot \alpha_3) \quad \text{or} \quad (x - s)(x^2 - \alpha_{1,\,4})(x^2 - \alpha_{2,\,3}),$$

where

$$\alpha_{i,\,j} = \alpha_i \cdot \alpha_j.$$

Expanding both sides in terms of powers of x, we have in the general case
for $\qquad\qquad\qquad\qquad n = 2m,$

$$
\begin{aligned}
x^{2m} &- x^{2m-1}\Sigma a_{11} + x^{2m-2}\Sigma\binom{12}{12} - \cdots - x \cdot \Sigma A_{11} + \Delta \\
&= x^{2m} - x^{2m-1}(s + s') + x^{2m-2}(s \cdot s' - \Sigma\alpha_{1,\,n-1}) + \cdots \\
&\quad + (-1)^{k-1}x^{2m-2k}(s \cdot s'\Sigma\alpha_{1,\,n-1} \cdot \alpha_{2,\,n-2} \cdots \alpha_{k-1,\,n-k+1} \\
&\quad - \Sigma\alpha_{1,\,n-1} \cdot \alpha_{2,\,n-2} \cdots \alpha_{k,\,n-k}) \\
&\quad + (-1)^{k-1}x^{2m-2k-1}(s+s')\Sigma\alpha_{1,\,n-1} \cdot \alpha_{2,\,n-2} \cdots \alpha_{k,\,n-k} + \cdots \\
&\quad + (-1)^{m-1}s \cdot s'\alpha_{1,\,n-1} \cdot \alpha_{2,\,n-2} \cdots \alpha_{m-1,\,n-m+1};
\end{aligned}
$$
(A)

for $\qquad\qquad\qquad\qquad n = 2m + 1,$

$$
\begin{aligned}
-x^n &+ x^{n-1}\Sigma a_{11} - x^{n-2}\Sigma\binom{12}{12} + \cdots x \cdot \Sigma A_{11} + \Delta \\
&= -x^n + x^{n-1}s + x^{n-2}\Sigma\alpha_{1,\,n} - \cdots \\
&\quad + (-1)^{k-1}x^{n-2k} \cdot \Sigma\alpha_{1,\,n-1} \cdot \alpha_{2,\,n-2} \cdots \alpha_{k,\,n-k} \\
&\quad + (-1)^{n-1}x^{n-2k-1} \cdot s \cdot \Sigma\alpha_{1,\,n-1} \cdot \alpha_{2,\,n-2} \cdots \alpha_{k,\,n-k} + \cdots \\
&\quad + (-1)^m \cdot s \cdot \alpha_{1,\,n-1} \cdot \alpha_{2,\,n-2} \cdots \alpha_{m,\,n-m};
\end{aligned}
$$
(B)

Where Σa_{11} represents the sum of the elements along the principal diagonal of the circulant Δ, $\Sigma\binom{12\,\cdots\,k}{12\,\cdots\,k}$, the sum of the coaxial minors of order k, and ΣA_{11} the sum of the coaxial minors of order $n - 1$.

Equating coefficients of like powers of x we have from (A)

$$(s + s')\Sigma\alpha_{1,\,n-1} \cdots \alpha_{k-1,\,n-k+1} = (-1)^{k-1}\Sigma\binom{12\,\cdots\,2k-1}{12\,\cdots\,2k-1},$$

$$s \cdot s'\Sigma\alpha_{1,\,n-1} \cdots \alpha_{k-1,\,n-k+1} - \Sigma\alpha_{1,\,n-1} \cdots \alpha_{k,\,n-k} = (-1)^{k-1}\Sigma\binom{12\,\cdots\,2k}{12\,\cdots\,2k},$$

$$(s + s')\Sigma\alpha_{1,\,n-1} \cdots \alpha_{k,\,n-k} = (-1)^k\Sigma\binom{12\,\cdots\,2k+1}{12\,\cdots\,2k+1},$$

that is,

$$(-1)^{k-1}s \cdot s'\frac{\Sigma\binom{12\,\cdots\,2k-1}{12\,\cdots\,2k-1}}{s + s'} - (-1)^k\frac{\Sigma\binom{12\,\cdots\,2k+1}{12\,\cdots\,2k+1}}{s + s'} = (-1)^{k-1}\Sigma\binom{12\,\cdots\,2k}{12\,\cdots\,2k}$$

or

$$s \cdot s' = \frac{(s + s')\Sigma\binom{12 \dots 2k}{12 \dots 2k} - \Sigma\binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma\binom{12 \dots 2k-1}{12 \dots 2k-1}}, \tag{1}$$

a constant ratio for all values of k from 1 to $m - 1$.

When $k = m$, the ratio becomes

$$\frac{\Sigma a_{11} \cdot \Delta}{\Sigma A_{11}} ;$$

from B,

$$\Sigma \alpha_{1, n-1} \cdots \alpha_{k, n-k} = (-1)^k \Sigma\binom{12 \dots 2k}{12 \dots 2k},$$

$$s \cdot \Sigma \alpha_{1, n-1} \cdots \alpha_{k, n-k} - (-1)^k \Sigma\binom{12 \dots 2k+1}{12 \dots 2k+1},$$

That is

$$s \cdot \Sigma\binom{12 \dots 2k}{12 \dots 2k} = \Sigma\binom{12 \dots 2k+1}{12 \dots 2k+1}$$

or

$$s = \frac{\Sigma\binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma\binom{12 \dots 2k}{12 \dots 2k}} . \tag{2}$$

In the case where $n = 2m$, we have

$$s' = \frac{\Sigma\binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma\binom{12 \dots 2k}{12 \dots 2k}} , \quad \text{if} \quad s = 0, \tag{1a}$$

It is known* that when $s = 0$ all the primary minors of Δ are equal. It follows, therefore, that the unique primary minor

$$A_1 = \frac{s' \cdot \Sigma\binom{12 \dots 2m-2}{12 \dots 2m-2}}{2m} . \tag{1b}$$

Similarly

$$s = \frac{\Sigma\binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma\binom{12 \dots 2k}{12 \dots 2k}} \quad \text{when} \quad s' = 0. \tag{1c}$$

If $s + s' = 0$, then

$$s^2 = s'^2 = \frac{\Sigma\binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma\binom{12 \dots 2k-1}{12 \dots 2k-1}} . \tag{1d}$$

* Borchardt, "Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante," *Crelle's Journal*, lvii, pp. 111–121. Cf. Muir, *Messenger of Mathematics*, New Series, No. 536, Vol. XLV, December, 1915.

If $s = s' = 0$, then

$$\Sigma\left(\begin{smallmatrix} 12\,\ldots\,2k+1 \\ 12\,\ldots\,2k+1 \end{smallmatrix}\right) = 0. \tag{1e}$$

From ($1b$) we see that, when $s = s' = 0$, the unique minor A_1 vanishes. In this case the determinantal equation contains only even powers of x and since all its roots are real it follows that the signs must be alternately positive and negative. That is, the signs of $\Sigma\left(\begin{smallmatrix}12\\12\end{smallmatrix}\right)$, $\Sigma\left(\begin{smallmatrix}1234\\1224\end{smallmatrix}\right)\cdots$ must be alternately negative and positive.

In the case of ($1d$) we see that the sums of the coaxial minors of odd order have the same sign.

In the case when $n = 2m + 1$ we have

$$\Sigma\left(\begin{smallmatrix} 12\,\ldots\,2k+1 \\ 12\,\ldots\,2k+1 \end{smallmatrix}\right) = 0 \quad \text{of} \quad s = 0. \tag{2a}$$

Here again the signs of the sums of coaxial minors of even order must be alternately negative and positive.

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Canada's Challenge

In Flanders Fields

By LIEUT.-COL. JOHN D. McCRAE*

In Flanders fields the poppies blow
   Between the crosses, row on row,
That mark our place, and in the sky
The larks, still bravely singing, fly,
Scarce heard amidst the guns below.
We are the dead. Short days ago
We lived, felt dawn, saw sunset glow,
Loved and were loved, and now we lie
   In Flanders fields.

Take up our quarrel with the foe!
To you from falling hands we throw
   The torch. Be yours to hold it high!
If ye break faith with us who die
We shall not sleep, tho' poppies grow
   In Flanders fields.

America's Answer

By R. W. LILLARD*

Rest ye in peace, ye Flanders dead.
The fight that ye so bravely led
We've taken up. And we will keep
True faith with you who lie asleep
With each a cross to mark his bed,
And poppies blowing overhead,
Where once his own life blood ran red.
So let your rest be sweet and deep
    In Flanders fields.

Fear not that ye have died for naught.
The torch ye threw to us we caught.
    Ten million hands will hold it high
    And Freedom's light shall never die!
We've learned the lesson that ye taught
    In Flanders fields.

*Written after the death of Lieut.-Col. McCrae, author of "In Flanders Fields," and printed in The New York Evening Post.

NEW BOOKS.

Go, Get 'Em! By William A. Wellman. Boston: The Page Co. Pp. 284. $1.50 net.

This is the story of a young American athlete who was the only American in the air over General Pershing's famous "Rainbow Division" when the Yankee troops made their first over-the-top attack on the Hun, and during that battle was in command of the lowest platoon of French fighting planes and personally disposed of two of the enemy's attacking aircraft.

His experience includes more than fighting above the troops. He was in Paris and Nancy during four distinct night-bombing raids by the Bosch and participated in rescues made necessary thereby. His story is very interesting.

The Yanks are Coming! By William S. McNutt. Boston: The Page Co. Pp. 267. $1.50 net.

This book will be interesting not only to those who have relatives in the war but to the public in general. He portrays the *spirit* of the American soldier and shows why he is so hard to defeat. It is a book which every American will be glad to read.

Carita. By Lucy M. Blanchard. Boston: The Page Co. Pp. 303. $1.50 net.

This is an interesting story of a young American girl who lived the early part of her life in Mexico with her parents and learned to like Mexico so well that it was not easy for her to return and become a good American. The book gives much interesting information of life in Mexico during the latter days of President Diaz.

The Man Who Won. By Leon D. Hirsch. Boston: The Page Co. Pp. 395. $1.50 net.

This is a novel which differs from most in that it is a story of political life with an interesting romance running through it.

The author tells how Edward Harrison, the political boss of the town, is forced to put aside his unscrupulous politics for a clean government. His disinherited son is called back from the West for this purpose. Through the book runs a charming story concerning Mrs. Harrison, a social climber, and Alice Lane, with whom the son falls in love.

Mortality Statistics. Washington: The Government Printing Office. Pp. 543.

This is a mine of statistics from the greater portion of the population of the United States.

Who's Who in America, 1918-1919. Chicago: A. N. Marquis & Co. Pp. 3296. $6.00.

"Who's Who in America" for the years 1918–1919 is just off the press and is more valuable than ever before. It contains more than 3,000 pages and nearly 23,000 life-sketches. There are 3,139 names in this volume which have appeared in no previous edition. Among the new names are hundreds which have gained prominence on account of the war, including those of many officers of the Army and Navy promoted from lower ranks, civilians appointed to offices in the National Army, and men and women selected to fill important civil commissions, created to assist in discharging the stupendous task on which the nation had entered.

This book is recognized the world over as the one indispensable reference book of contemporary American biography, and has its established place in homes, schools and business offices, as well as in the public library and the newspaper office. It is used constantly in all departments of the government at Washington.

Every sketch is a life-story in a nutshell and entirely dependable. The book is a marvel of comprehensiveness and is the result of twenty years of unremitting labor. It was established in 1898 by Albert Nelson Marquis, and has been under his editorial supervision ever since. With publication of this edition—the tenth biennial issue—it celebrates its twentieth anniversary.

The volume contains practically all the names of those whose position or achievements make them of general interest. The sketches are arranged in alphabetical order, making it easy to find the particular information desired. The geographical index alone, by states, cities and towns, comprises 120 pages.

Business Arithmetic. By C. W. Sutton and N. J. Lennes. New York: Allen and Bacon. Pp. xiv + 466.

The first thirteen chapters of this book are a review of the fundamental operations with integers, fractions and decimal fractions, with much drill work for speed and accuracy, and a good deal of emphasis on checking results.

The remainder of the book is given up to the usual business topics, though the more advanced subjects, such as cost accounting, are purposely excluded.

Among the good features of the book are the pages of drill and review at the end of each chapter.

Le Premier Livre. By Albert A. Meras and B. Meras. New York: American Book Co. Pp. 200.

This combined reader and grammar is supposed to cover one half year's work. It aims to give the pupils from the very beginning interesting and practical French, and so centers all the work about Hector Malot's story "Sans Famille." The grammar, conversation and com-

position, which are given in each lesson, are based on consecutive parts
of this story.

The book seems well written and practical, as well as unusually in-
teresting.

Freshman English. By MARY EVELYN SHIPMAN. Boston: D. C. Heath
and Co. Pp. 46.

The author presents some very interesting and stimulating viewpoints
in this little booklet on teaching English in the first year of college.

The argument is based on the fact that the usual course in freshman
composition repeats considerable of the work done in high school and
so loses the interest with which the pupil would naturally approach a
college subject for the first time. The author shows how the course
can instead be adapted to the interests and needs of the student, while
the preceding work is fixed by its use in the new field.

Commercial Algebra, Book II. By GEORGE WENTWORTH, DAVID EUGENE
SMITH, and WILLIAM S. SCHLAUCH. Boston: Ginn & Co. Pp. 250.
Price $1.12.

The first book of this set was reviewed in a former number. The
second book, which is intended for advanced classes in commercial high
schools, goes much more deeply into the subject. It also derives its
problems more largely from actual business experiences.

The subject matter includes logarithms and the slide rule, compound
interest and its application, equation of payments, life insurance, and
several other topics.

This should prove a very valuable book for classes studying such
topics. Mr. Schlauch is one of the best informed men in the country
in all that pertains to the mathematics of business, and his knowledge
and enthusiasm show clearly in this series.

Junior High School Mathematics, Book III. By GEORGE WENTWORTH,
DAVID EUGENE SMITH, and JOSEPH CLIFTON BROWN. Boston: Ginn &
Co. Pp. 282. Price 96 cents.

This is the last book of this Junior High School Series, of which
Books I and II have already been reviewed.

This volume, designed for the ninth school year, reviews the earlier
algebra, so that it can be used by pupils who have not used the first
two books; then continues this subject, confining itself largely to the
parts most needed for application.

After 112 pages it takes up trigonometry, and gives 22 pages to the
study of the use of functions in solving the right triangle.

The remainder of the book is demonstrative geometry, the usual first
book and most of the area theorems being treated.

It is the aim of the authors of this series not only to give a pupil
taking the junior high school course a working knowledge of the funda-

mentals of algebra, geometry and trigonometry, but also to have given him such an insight into the subjects that he can judge of their value to him, and know whether or not to continue them.

The Course in Science, Vol. V, Francis W. Parker School Year Book. Pp. 168. Francis W. Parker School, Chicago.

This issue of the Year Book presents the science work as taught in the Francis W. Parker School, throughout both the elementary and high school grades. It is the result of a number of years of independent, experimental, and developmental work on the part of many members of the faculty, and is an attempt to improve the choice of materials, to suggest better methods of presentation, and to unify the science instruction of the school.

Following a presentation of the general principles underlying the organization of the course, the detailed outlines are given grade by grade and course by course, showing how all the work in science may be based upon the interests, activities and problems of the pupil. Not only is the course given in outline, but the outcome is indicated by many examples of the pupil's work, as shown by their own papers, or as given in morning exercises. The experimental work is fully presented, together with many references for class reading or as aids to the teacher.

The book is well illustrated, and should be of interest to all teachers in the elementary school, to high school teachers of science, and to principals and superintendents interested in the making of a vital school curriculum based upon the interests and activities of the children.

The Sandman: His Bunny Stories. By HARRY W. FREES. Boston: The Page Company. Pp. viii + 274. Price $1.50.

These are excellent stories for young children. They are short enough to hold the interest, and they point morals without preaching. The illustrations are very clever photographs of rabbits acting out the scenes of the stories.

The Boy Scouts of Kendalville. By BREWER CORCORAN. Pp. 270. Price $1.50.

The author has told a good clean story of adventure and patriotism in a way to thrill any boy who reads it. It is an excellent book for boys of eleven years or over.

The Sandman: His Indian Stories. By W. S. PHILLIPS. Boston: The Page Company. Pp. xvi + 292. Price $1.50.

This is a collection of delightfully told Indian stories and legends, written by one who knows the Indian at first hand, and who shows in this book his sympathy with, and understanding of them.

The book is well adapted for use by children of quite a wide range in age, and seems particularly well fitted for use as a supplementary school book on Indian life.

Three Boys in the Indian Hills. By W. S. PHILLIPS. Boston: The Page
Company. Pp. x + 326. Price $1.50.

This is a story of the northern plains at the time when it was peopled
by Indians, and game of various kinds was abundant. It does not over-
emphasize the pleasure of a wild life, but tells an interesting tale in a
historical setting, and so gives much information about the life of that
time and country.

A Handbook of American Private Schools. 1918 Edition. By PORTER E.
SARGENT. Boston. Pp. 711.

This book is not only a complete and very accurate directory of schools
and their executives, but also an encyclopedia of the information that is
necessary for running a school, and for keeping in touch with educa-
tional movements.

There are lists of schools of every kind, ranging from the usual day
or boarding schools to those for various special purposes. The schools
are discussed in well organized comparative tables, as well as in sepa-
rate paragraphs. Summer camps are also thoroughly covered.

For the school principal, one of the most useful features of the book
is its directories of educational associations, periodicals, agencies, pub-
lishers, firms dealing in different kinds of supplies and equipment, etc.
The brief but definite article on developments in education, and the
bibliography of recent educational books are also very valuable.

All in all the book is by far the most comprehensive and the most
intelligently and fairly written of any that have attempted to approxi-
mate its purpose. It should be in the hands of every school executive,
and its facts about schools are very useful for the colleges also. One
of its greatest values, however, should be in its help to parents, as it
gives in usable form the information necessary to find the school best
fitted for a child's needs.

Geometry for Junior High Schools. By WILLIAM BETZ. Published by the
Board of Education, Rochester, N. Y. Pp. 112. Price 70 cts.

This book is the result of experimentation in the Washington Junior
High School of Rochester, N. Y. It embodies many of the ideas of its
author as they have been expressed in his talks and articles, a number of
which have been published in THE MATHEMATICS TEACHER. Mr. Betz
believes both in motivation by practical applications and in teaching a
pupil to observe and to reason. His object in this work is to give an
introductory geometrical course that will acquaint the pupil with geo-
metric forms and important facts about them, and it naturally follows
that such a book from him contains many features making it worthy of
consideration for Junior High School use.

NOTES AND NEWS.

SUBSCRIBERS and members should be prompt in sending in their renewal subscriptions as soon as they are due. Postal regulations will not permit delay.

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THERE has just been issued by the Bureau of Education at Washington a bulletin on The Training of Teachers of Mathematics for Secondary Schools of the Countries Represented in the International Commission on the Teaching of Mathematics. This bulletin has been prepared by Professor R. C. Archibald, of Brown University. It is a work of nearly three hundred pages, giving in great detail the requirements set by the various governments for a teacher of secondary mathematics. The Bureau of Education has a limited number of copies of this bulletin which it can send to those who are particularly interested in the work. After this limited number has been exhausted, copies can be obtained from the Superintendent of Documents, Government Printing Office at Washington, D. C., at 30 cents per copy.

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OLD ENGLAND KEEPS HER HEAD!

Detailed reports from London regarding the great new Education Act which Parliament has passed, remodelling from the ground up England's entire educational system, show one striking omission, an omission upon which Americans may well ponder:

The Act contains no provision whatsoever for compulsory military training.

From one point of view it is a magnificent tribute to the sound English spirit. Even in the midst of war, with Germany a stone's throw away, England can study her educational problems coolly and decide quietly to keep her schools as training centers

101

for *individualism plus service,* rather than, in a panic, to sacrifice them to the Prussian system of military drill.

And it is all the more striking because England, like America, has had various strenuous organizations dedicated to the job of fastening military training upon the school system. They have been well financed and have held meetings and distributed literature showing the horrors of life without military training; they have had questions "put" in the House and, in general, have betrayed a fine zeal on behalf of their propaganda. But H. A. L. Fisher, the Minister of Education, told a delegation from the Miners' Federation some months ago, that the government had canvassed the question of compulsory drill for the secondary schools and had decided that *the innovation had neither educational nor military value and would not be adopted.*

Mr. Fisher has proved as good as his word. The Education Act, which sweeps out of existence eleven educational acts and repeals parts of eleven others, provides for comulsory education up to fourteen years. Between the ages of fourteen and eighteen all English boys and girls must attend either the regular schools or, if they are obliged to work, then they must attend continuation schools and their employers must help to make that school attendance possible. Physical training is provided without stint but of military training there is not to be a trace.

England certainly has a quality all her own!

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"New Jersey Says: 'No!'"

For the epidemic of military training which is sweeping the school boards of the country there is probably nothing to be done save to wait until it has run its course. It is partly due to a recognition that in a prolonged war the sixteen and seventeen year old boys might eventually be called to the colors, but chiefly it seems to us to be a form of vicarious patriotism for the elderly gentlemen on the school boards and for the local newspaper editors—those robust guardians of public virtue and the common schools. Public opinion itself is by no means hostile to the innovation.

But what the school boards do today they can undo tomorrow, and it is the business of the teaching profession to make sure that when the epidemic has run its course, we shall not be *permanently* saddled with compulsory military training in the high schools of the country. They have too much real work to do to be shackled with the bad pedagogy of military drill.

And it is significant that there is at least one state in the Union which seems to be relatively free from the craze for juvenile military drill. President Wilson's own state of New Jersey has apparently kept its head through these times, thanks to the moderate, sensible and utterly convincing report on the question of compulsory military drill in the secondary schools made by the commission appointed two years ago at the instance of the New Jersey legislature. That report, signed by a commission which included a member of the New Jersey National Guard, disposes effectually of the wild and wooly claims put forth for military drill for growing boys. It has been reprinted for free distribution by the American Union Against Militarism, Westory building, Washington, D. C., and ought to be in the hands of every teacher and school superintendent in the country for the day when it can be used to restore the schools to their normal course of development.

* * * *

THIS Christmas must count as no Christmas of recent years has counted. The spirit of Christmas must be kept up. Only sensible, wisely selected things can be given, and one gift should provide for many. Here it is—an ideal gift, for one and the whole family are sure to be delighted with it. *The Youth's Companion* fills the bill completely, coming all new 52 times a year. Stories, articles, receipts, special pages and more in quantity for all ages than any monthly magazine gives in a year. A distinct benefit to all hands. You give cheer, uplift, inspiration and entertainment—an actual need of these times. The *Companion* is still only $2.00 a year.

Don't miss Grace Richmond's great serial, Anne Exeter, ten chapters, beginning December 12.

The following special offer is made to new subscribers: (1) *The Youth's Companion*—52 issues of 1919. (2) All the re-

maining weekly issues of 1918.   (3) The Companion Home Calendar for 1919.   All the above for only $2.00, or you may include (4) McCall's Magazine—twelve fashion numbers.   All for only $2.50.   The two magazines may be sent to separate addresses if desired.   *The Youth's Companion,* Commonwealth Ave. & St. Paul St., Boston, Mass.   New Subscriptions Received at this Office.

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